

A Cahn-Hilliard equation with singular diffusion

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Abstract

In the present work, we address a class of Cahn-Hilliard equations characterized by a singular diffusion term. The problem is a simplified version with constant mobility of the Cahn-Hilliard-de Gennes model of phase separation in binary, incompressible, isothermal mixtures of polymer molecules. It is proved that for any final time T , the problem admits a unique energy type weak solution, defined over $(0, T)$. For any $\tau > 0$ such solution is classical in the sense of belonging to a suitable Hölder class over (τ, T) , and enjoys the property of being separated from the singular values corresponding to pure phases.

Key words: Cahn-Hilliard equation, singular diffusion, variational formulation, existence theorem.

AMS (MOS) subject classification: 35K35, 35K67, 35A01, 82D60.

1 Introduction

This paper is devoted to the mathematical analysis of the following class of parabolic systems:

$$u_t - \Delta w = 0, \quad (1.1)$$

$$w = -a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) - \lambda u + \varepsilon u_t, \quad (1.2)$$

on $(0, T) \times \Omega$, Ω being a bounded smooth subset of \mathbb{R}^3 and $T > 0$ an assigned final time. The system is coupled with the initial and boundary conditions

$$u|_{t=0} = u_0, \quad \text{in } \Omega, \quad (1.3)$$

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = 0, \quad \text{on } \partial\Omega, \quad \text{for } t \in (0, T) \quad (1.4)$$

and represents a variant of the Cahn-Hilliard model for phase separation in binary materials. Here, λ and ε are nonnegative parameters, where the case $\varepsilon > 0$ accounts for a viscosity effect that may appear the frame of Cahn-Hilliard models (see, e.g., [15]). Moreover, the function f stands for the derivative of the so-called logarithmic potential

$$F(r) = (1 - r) \log(1 - r) + (1 + r) \log(1 + r), \quad r \in [-1, 1], \quad (1.5)$$

namely,

$$f(r) = \log(1+r) - \log(1-r) = \log \frac{1+r}{1-r}, \quad r \in (-1, 1). \quad (1.6)$$

Finally, we assume that the function a has the form

$$a(r) = \frac{2}{1-r^2} = f'(r). \quad (1.7)$$

The initial-boundary value problem given by (1.1)-(1.3) will be noted as Problem (P) in the sequel.

The above problem is a simplified version (assuming constant mobility and normalized physical quantities) of the Cahn-Hilliard-de Gennes model [9] of phase separation in binary, incompressible, isothermal mixtures of polymer molecules of different types $i = 1$ and $i = 2$, quenched below a critical temperature.

Following the Cahn-Hilliard-de Gennes theory [9], [10] (see also [3], [21], [25]), we describe now briefly the physical basis of the presented model. Each molecule i in the mixture consists of N_i segments of size σ_i (so-called lattice constant) with the quantity $R_{g_i}^2 = \frac{1}{6}N_i\sigma_i^2$ denoting the mean radius of gyration of the i -th polymer molecule. The variables u and w in Problem (P) have the meaning of the rescaled order parameter and the exchange chemical potential. For clarity of physical description let us use just now in place of $u \in [-1, 1]$ the original variable $\chi = \chi_1 \in [0, 1]$, denoting the volume fraction of component $i = 1$. Then, by the incompressibility condition, $\chi_1 + \chi_2 = 1$ everywhere in the sample. The variable

$$u := 2\chi - 1 \in [-1, 1] \quad (1.8)$$

is introduced for the sake of mathematical convenience.

The model is governed by the Flory-Huggins-de Gennes free energy functional [9] which in the isothermal case is

$$\mathcal{F}_{FHdG}(\chi) = \int_{\Omega} \left(F_{FH}(\chi) + \frac{1}{2}a(\chi)|\nabla\chi|^2 \right). \quad (1.9)$$

The homogeneous (volumetric) free energy $F_{FH}(\chi)$ has the Flory-Huggins form

$$F_{FH}(\chi) = \frac{1}{N_1}\chi \log \chi + \frac{1}{N_2}(1-\chi) \log(1-\chi) + \lambda\chi(1-\chi), \quad (1.10)$$

where λ , called the Flory-Huggins interaction parameter, measures the strength of interaction between two kinds of species. The second term in the integrand (1.9) gives the weighted contribution due to composition gradients. By de Gennes theory [9], it has the characteristic singular form

$$\frac{1}{2}a(\chi) = \frac{1}{6} \left(\frac{R_{g_1}^2}{N_1\chi} + \frac{R_{g_2}^2}{N_2(1-\chi)} \right) = \frac{1}{36} \left(\frac{\sigma_1^2}{\chi} + \frac{\sigma_2^2}{1-\chi} \right) \quad (1.11)$$

which is due to the connectivity of the chains that constitute the polymer molecules. In the symmetrical case $N_1 = N_2 = N$, $\sigma_1 = \sigma_2 = \sigma$, $R_{g_1}^2 = R_{g_2}^2 = R_g^2$, the expression (1.11) simplifies to

$$\frac{1}{2}a(\chi) = \frac{\sigma^2}{36} \frac{1}{\chi(1-\chi)}. \quad (1.12)$$

Here we note that the singular form of $a(\chi)$ introduces an infinite energy penalty near the pure phases. This fact turns out to have an important mathematical consequence related to the separation property of the solution (see (2.23) and Remark (2.6) below).

Under the incompressibility condition, $\chi_1 + \chi_2 = 1$, the conservation of mass for component 1 is given by

$$\chi_t + \nabla \cdot j = 0, \quad j = -\Lambda(\chi)\nabla w. \quad (1.13)$$

Here j is the mass flux defined as the product of the Onsager diffusion coefficient $\Lambda(\chi)$ (effective mobility) and the gradient of the exchange potential w which is the difference between the chemical potentials of the components, $w = w_1 - w_2$. Shortly, we shall refer to Λ and w as the mobility and the chemical potential, respectively.

According to the derivations in [9], [3], [5], the expression for Λ takes the form

$$\Lambda(\chi) = \frac{\Lambda_1 \Lambda_2}{\Lambda_1 + \Lambda_2} = \chi(1 - \chi)\Lambda_0, \quad (1.14)$$

where $\Lambda_i = \chi_i \Lambda_0$ is the Onsager coefficients for the i -th component and Λ_0 is a positive constant.

We point out that the concentration dependence of the mobility Λ is also typical for phase separation of small molecule systems, described by the classical Cahn-Hilliard equation [6], [7]; see also [12], [14], [20] and the references therein.

The chemical potential w is defined as the first variation of the functional (1.9) yielding the following equivalent expressions:

$$\begin{aligned} w &= \frac{\delta \mathcal{F}_{FHdG}}{\delta \chi} \\ &= F'_{FH}(\chi) + \frac{1}{2} a'(\chi) |\nabla \chi|^2 - \operatorname{div}(a(\chi) \nabla \chi) \\ &= F'_{FH}(\chi) - \frac{1}{2} a'(\chi) |\nabla \chi|^2 - a(\chi) \Delta \chi \\ &= F'_{FH}(\chi) - \sqrt{a(\chi)} \operatorname{div}(\sqrt{a(\chi)} \nabla \chi). \end{aligned} \quad (1.15)$$

As in the standard Cahn-Hilliard theory, the chemical potential can include an additional term $\varepsilon \chi_t$, with a positive constant ε , accounting for possible viscous effects.

The equations (1.13), (1.14), and (1.15) with the term $\varepsilon \chi_t$, $\varepsilon \geq 0$, lead to the following degenerate singular Cahn-Hilliard-de Gennes polymer system:

$$\chi_t - \Lambda_0 \operatorname{div}(\chi(1 - \chi) \nabla w) = 0, \quad (1.16)$$

$$w = -a(\chi) \Delta \chi - \frac{a'(\chi)}{2} |\nabla \chi|^2 + F'_{FH}(\chi) + \varepsilon \chi_t \quad (1.17)$$

to be considered together with the initial and appropriate boundart conditions.

As a first step in the study of such system we consider the case of constant mobility by linearizing the principal part $\chi(1 - \chi)$ in (1.16). Moreover, for mathematical convenience, we replace χ by the variable u defined by (1.8), restrict ourselves to the symmetrical mixture (1.12), and set all physical constants equal to unity. This leads to (1.1)-(1.2).

Apart from the well-known Cahn-Hilliard-de Gennes model described above, we mention also recently developed two-fluids models for viscoelastic phase separation in polymer solutions, see [29] and the references therein.

It is known that the process of polymer phase separation is important both for its theoretical aspect and due to unusual morphology for specific materials applications. On the contrary to extensive physical and numerical literature on polymer mixtures (for review see e.g., [3], [10], [21], [23], [1]), mathematical aspects of the corresponding models have not been so far addressed sufficiently. The structure of steady-state solutions to degenerate singular polymer system (1.16)-(1.17) has been analysed by Mitlin et al. [18], [17], and Witelski [28]. The existence of weak solutions to the Cahn-Hilliard equation with logarithmic potential $F(u)$, degenerate mobility $\Lambda(u) = 1 - u^2$, and constant gradient coefficient $a > 0$ has been studied by Elliott and Garcke [12]; we refer also to [8] and [14] for further study of degenerate problems. For the standard Cahn-Hilliard equation with logarithmic potential $F(u)$ and constant coefficient $a > 0$, optimal regularity of weak solutions is analyzed in [19], where in particular the separation property (2.23) is obtained in space dimensions 1 and 2. The existence and uniqueness of weak solutions to the Cahn-Hilliard equation with logarithmic potential and nonlinear positive, bounded coefficient $a(u)$ has been recently proved by the authors in [26].

We mention also a closely related sixth order Cahn-Hilliard type problem with nonlinear coefficient $a(u)$, considered recently in [26] for a singular (e.g., logarithmic) potential, and in [24] for a polynomial potential. As a special case, in [26] the behavior of the solutions when the sixth order term is let tend to zero was analysed.

Finally, we point out that the theoretical investigation of polymer models was initiated by Alt and the second author in [1], where general nonisothermal phase transition models with a conserved order parameter have been derived and, in particular, polymer free energy models have been presented

along with an extensive list of references. The authors of [1] have obtained also some partial, unpublished results [2] on the existence of weak solutions to degenerate singular polymer model (1.16)-(1.17) by applying the methods due to Elliott and Garcke [12], and Elliott and Luckhaus [13]. This unsolved problem has become the motivation of the present study which uses a different approach developed previously in [26].

As already mentioned, as a first step of the analysis we assume that the mobility is constant, and extend our methods applied in [26] in the case of a nonlinear (but bounded) coefficient $a(u)$ to singular $a(u)$.

There are two main ideas behind our approach. The first one, standard in the analysis of Cahn-Hilliard systems, exploits the characteristic variational structure of the system (1.1)-(1.2). The second non-standard one consists in introducing appropriate changes of variables.

The variational structure becomes evident by (formally) testing (1.1) by w , (1.2) by u_t , taking the difference of the obtained relations, integrating with respect to space variables, using the *no-flux* conditions (1.4), and performing suitable integrations by parts. Then one readily gets the *a-priori* bound

$$\frac{d}{dt}\mathcal{E}(u) + \|\nabla w\|_{L^2(\Omega)}^2 + \varepsilon\|u_t\|_{L^2(\Omega)}^2 = 0, \quad (1.18)$$

which has the form of an *energy equality* for the *energy functional*

$$\mathcal{E}(u) = \int_{\Omega} \left(\frac{a(u)}{2} |\nabla u|^2 + F(u) - \lambda \frac{u^2}{2} \right), \quad (1.19)$$

where the interface (gradient) part contains the nonlinear function a . In other words, the system (1.1)-(1.2) arises as the $(H^1)'$ -gradient flow problem for the functional \mathcal{E} .

However, the energy estimate (1.18) is not sufficient to obtain existence of a solution to (1.1)-(1.2) via approximation-compactness methods. Actually, to apply this strategy, one also needs some control of the second space derivatives of u and of the singular coefficients a and a' in (1.2). To obtain this, two changes of variables will play an important role. The first one, motivated by the structure of the fourth formula in (1.15) and applied also in [26], consists in introducing the variable z such that $\nabla z = \frac{1}{\sqrt{2}} \sqrt{a(u)} \nabla u$. Then, the Laplacean of z can be (formally) estimated simply by testing the equivalent z -formulation of (1.2) (namely, (2.13) below) by $-\Delta z$. This gives the desired control on second space derivatives.

On the other hand, even in the equivalent formulation (2.13), one has to control a coefficient (namely, $\phi'(u)$), that explodes polynomially fast as $|u|$ approaches 1. This is a nontrivial issue since the nonlinear term $f(u)$ in (1.2) (or in (2.13)), for which is relatively simple to get a L^2 -control, explodes only *logarithmically* fast (and, hence, an L^p -estimate of $f(u)$ would help to control $\phi'(u)$, or $a(u)$, only for $p = \infty$). To overcome this further difficulty, a second additional change of variable comes into help. Namely, we set $v := f(u)$, which represents the monotone part of the volumetric chemical potential. We then see that the formulation of equation (1.2) in terms of v (namely, (2.17) below) does no longer contain singular coefficients of polynomial type; it presents, however, the cubic term $u|\nabla v|^2$. To control it, we use techniques based on entropy-type estimates (cf. [8]), with a rather careful and *ad-hoc* choice of test functions (cf. (4.40) and (4.63) below). In this way we can both prove an L^∞ -bound for v (and consequently the “separation property” (2.23)), for strictly positive times, and also control the cubic term $u|\nabla v|^2$ starting from the initial time $t = 0$. This is the key step that permits to get existence of a weak solution (for the v -formulation of the system) for initial data u_0 that have only the natural energy regularity (i.e., such that $\mathcal{E}(u_0) < +\infty$).

Regarding additional properties of solutions, we remark that, as already noticed in [11], the energy (1.19) with the coefficient a given by (1.7) is convex with respect to u (up to the lower order λ -perturbation). This basic property permits to prove parabolic time-regularization properties of weak solutions, as well as uniqueness, in a relatively standard way.

The plan of the paper is as follows. In the next Section 2 we will present the main assumptions and give the statement of our main results. In Section 3, we will prove local existence and uniqueness of a strong (i.e., lying in a suitable Hölder class) solution. The main *a-priori* estimates needed in the proof of global existence will be detailed in the subsequent Section 4. On the basis of these estimates, in the final Section 5 we shall show global existence, uniqueness, and time-regularization properties of weak solutions.

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2 Notation and main results

Let Ω be a smooth bounded domain of \mathbb{R}^3 of boundary Γ , $T > 0$ a given final time, and let $Q := (0, T) \times \Omega$. Let $H := L^2(\Omega)$, endowed with the standard scalar product (\cdot, \cdot) and the norm $\|\cdot\|$. Let also $V := H^1(\Omega)$. We identify H with H' so that the chain of continuous embeddings $V \subset H \subset V'$ holds. We indicate by $\langle \cdot, \cdot \rangle$ the duality between V' and V and by $\|\cdot\|_X$ the norm in the generic Banach space X . We note as A the weak Laplace operator with no-flux boundary conditions, namely

$$A : V \rightarrow V', \quad \langle Av, z \rangle := \int_{\Omega} \nabla v \cdot \nabla z, \quad \forall v, z \in V. \quad (2.1)$$

We also set

$$W := \{z \in V : Az \in H\} = \{z \in H^2(\Omega) : \partial_n z = 0 \text{ on } \Gamma\}, \quad (2.2)$$

which is a closed subspace of $H^2(\Omega)$. In all what follows we shall assume that F , f and a are given, respectively, by (1.5), (1.6) and (1.7). Moreover, we will assume that u_0 is an initial datum having *finite energy* \mathcal{E} (cf. (1.19)), namely

$$\mathbb{E}_0 := \mathcal{E}(u_0) < +\infty. \quad (2.3)$$

It is worth noting that, since F has the expression (1.5), the above is equivalent to asking

$$-1 \leq u_0 \leq 1 \text{ a.e. in } \Omega, \quad a^{1/2}(u_0) \nabla u_0 \in H. \quad (2.4)$$

Moreover, due to (1.7), if u_0 satisfies (2.4) then $u_0 \in V$, in particular. Letting, for a generic summable function $v : \Omega \rightarrow \mathbb{R}$, v_{Ω} denote its spatial mean value, we will also assume that

$$m := (u_0)_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u_0 \in (-1, 1). \quad (2.5)$$

In other words, we cannot admit the case when u_0 coincides with $+1$ (or with -1) almost everywhere in Ω . This is a standard assumption when dealing with Cahn-Hilliard systems containing constraint terms (cf, e.g., [16] for more details).

Let us note also that, substituting the expression (1.7) for a , (1.2) can be rewritten as

$$w = -\frac{2}{1-u^2} \Delta u - \frac{2u}{(1-u^2)^2} |\nabla u|^2 + f(u) - \lambda u + \varepsilon u_t. \quad (2.6)$$

It is now convenient to introduce a couple of additional variables which permit to give alternative formulations of (2.6). To start with, we compute some derivatives of a . From (1.5) and (1.7), we have

$$a'(r) = f''(r) = \frac{4r}{(1-r^2)^2}, \quad (2.7)$$

as well as

$$a''(r) = f'''(r) = \frac{4(1+3r^2)}{(1-r^2)^3}. \quad (2.8)$$

In the sequel, for a generic locally integrable real-valued function ψ defined in an open neighbourhood of 0, we will write

$$\widehat{\psi}(r) := \int_0^r \psi(s) \, ds. \quad (2.9)$$

Then, of course, we have

$$\widehat{a}(r) = \int_0^r a(s) \, ds = f(r) = \log(1+r) - \log(1-r). \quad (2.10)$$

Next, we introduce

$$\phi(r) := \frac{\sqrt{2}}{2} \int_0^r a^{1/2}(s) \, ds = \int_0^r \frac{ds}{(1-s^2)^{1/2}} = \arcsin r. \quad (2.11)$$

Then, we notice that, setting

$$z := \phi(u) = \arcsin u, \quad (2.12)$$

equation (1.2) can be rewritten in the equivalent form

$$w = -2\phi'(u)\Delta z + f(u) - \lambda u + \varepsilon u_t. \quad (2.13)$$

Next, we put $v := f(u)$. Then, a simple computation gives

$$u = f^{-1}(v) = \frac{e^v - 1}{e^v + 1} =: j(v). \quad (2.14)$$

Moreover, (1.2) can be rewritten as

$$w = -\Delta v + v + \frac{a'(u)}{2} |\nabla u|^2 - \lambda u + \varepsilon u_t. \quad (2.15)$$

Noting that

$$\frac{a'(u)}{2} |\nabla u|^2 = \frac{2u}{(1-u^2)^2} |\nabla u|^2 = \frac{u}{2} |f'(u) \nabla u|^2, \quad (2.16)$$

we finally obtain from (2.14)

$$w = -\Delta v + v + \frac{j(v)}{2} |\nabla v|^2 - \lambda j(v) + \varepsilon u_t = -\Delta v + v + \frac{1}{2} \frac{e^v - 1}{e^v + 1} |\nabla v|^2 - \lambda j(v) + \varepsilon u_t. \quad (2.17)$$

In the sequel, we shall indicate by c a generic positive constant, whose value may vary on occurrence, allowed to depend on the parameters of the system (more precisely, on the functions a and f , on λ and on Ω), and, in particular, not on approximating parameters. Moreover, the constants c will not be allowed to depend on the choice of initial data. However, they may depend on the prescribed mean value m . The notation κ will be used for positive constants (depending on the same quantities as c) appearing in estimates from below. We will also use the notation $Q(\cdot)$ (or $Q(\cdot, \cdot)$), with Q indicating a computable function with values in $[0, +\infty)$, increasingly monotone in each of its argument, whose expression can depend on the same quantities as c . For instance, the expression $Q(\mathbb{E}_0)$ will stand for a monotone function of the “initial energy” \mathbb{E}_0 .

We can now introduce the concepts of “classical” and of “weak” solution needed in the subsequent analysis.

Definition 2.1. A “strong”, or “classical”, solution to Problem (P) over the time interval $(0, T)$ is a couple (u, w) with the regularity

$$u \in W^{1,\infty}(0, T; V') \cap H^1(0, T; V) \cap L^\infty(0, T; H^2(\Omega)), \quad \varepsilon u \in W^{1,\infty}(0, T; H), \quad (2.18)$$

$$w \in L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega)), \quad \varepsilon w \in L^\infty(0, T; H^2(\Omega)), \quad (2.19)$$

satisfying, a.e. in $(0, T)$, the equations

$$u_t - \Delta w = 0, \quad \text{a.e. in } \Omega, \quad (2.20)$$

$$w = -a(u)\Delta u - \frac{a'(u)}{2} |\nabla u|^2 + f(u) - \lambda u + \varepsilon u_t, \quad \text{a.e. in } \Omega, \quad (2.21)$$

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = 0, \quad \text{a.e. on } \Gamma, \quad (2.22)$$

together with the initial condition $u|_{t=0} = u_0$ and, for all $(t, x) \in [0, T] \times \overline{\Omega}$, the separation property

$$-1 + \epsilon \leq u(t, x) \leq 1 - \epsilon, \quad \text{for some } \epsilon > 0. \quad (2.23)$$

Remark 2.2. Thanks to (2.23), the component u of any “classical” solution to Problem (P) is uniformly separated from the singular values ± 1 of a and f . Hence, by applying the standard theory of quasilinear parabolic equation and a bootstrap argument, we can see that (u, w) is in fact smoother, with its regularity being limited only by the regularity of the initial datum. In other words, at least for times $t > 0$, a classical solution can be thought to be arbitrarily regular.

Definition 2.3. A “weak”, or “energy”, solution to Problem (P) over the time interval $(0, T)$ is a couple (u, w) with the regularity

$$u \in H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^\infty((0, T) \times \Omega), \quad \varepsilon u \in H^1(0, T; H), \quad (2.24)$$

$$F(u) \in L^\infty(0, T; L^1(\Omega)), \quad (2.25)$$

$$v = f(u) \in L^2(0, T; V), \quad \nabla v \in L^p((0, T) \times \Omega) \text{ for some } p > 2, \quad (2.26)$$

$$w \in L^2(0, T; V), \quad (2.27)$$

satisfying, a.e. in $(0, T)$, the equations

$$u_t + Aw = 0, \quad \text{in } V', \quad (2.28)$$

$$w = -\Delta v + v + \frac{u}{2} |\nabla v|^2 - \lambda u + \varepsilon u_t, \quad \text{a.e. in } \Omega, \quad (2.29)$$

$$v = f(u), \quad \text{a.e. in } \Omega, \quad (2.30)$$

$$\partial_n v = 0, \quad \text{a.e. on } \Gamma. \quad (2.31)$$

together with the initial condition $u|_{t=0} = u_0$.

We can now state our main result, regarding existence, uniqueness, and regularization properties of weak solutions to Problem (P):

Theorem 2.4. Let f and a be given by (1.6), (1.7). Let u_0 satisfy (2.3) and (2.5). Finally, let Ω be convex. Then, for any $T > 0$, Problem (P) admits at least a weak solution (u, w) defined over $(0, T)$. Moreover, for any $\tau > 0$, (u, w) is a “classical” solution over (τ, T) . In particular, the separation property (2.23) holds on (τ, T) with

$$\epsilon^{-1} = Q(\mathbb{E}_0, \tau^{-1}). \quad (2.32)$$

Finally, uniqueness holds in the class of weak solutions that are classical for strictly positive times.

For strictly positive times, (1.2) can be interpreted in any of the equivalent formulations (2.6), (2.13), or (2.17); actually, u is a classical solution for $t > 0$. On the other hand, when looking at the behavior near $t = 0$, it is crucial to view (1.2) in the form (2.29) which appears to be the only formulation permitting to take the approximation limit $\delta \searrow 0$ starting from the initial time.

Remark 2.5. If we have in addition that the initial datum satisfies $u_0 \in H^2(\Omega)$ with $u_0(x) \in [-1 + \epsilon_0, 1 - \epsilon_0]$ for all $x \in \overline{\Omega}$ and some $\epsilon_0 \in (0, 1)$, there are no complications due to the boundary layer $t \searrow 0$, and so u can be seen as a “classical” solution over the whole $(0, T)$. This can be deduced simply by using the classical Gronwall lemma (instead of the uniform Gronwall lemma) in the a-priori estimates. As a consequence, estimates (4.30), (4.60) and (4.62) below hold in fact with $\tau = 0$ in this case.

Remark 2.6. It is worth stressing that the separation property (2.23) holding for our model is instead an open issue in the frame of the standard Cahn-Hilliard system with logarithmic nonlinearity $f(u)$, at least in the three-dimensional setting (cf. [19] for further remarks).

3 Local strong solutions

In this section, we will prove existence of at least one *local in time* classical solution to Problem (P). With this aim, we first introduce a regularization of the initial datum u_0 . This is the object of the following

Lemma 3.1. *Let u_0 satisfy (2.3) and (2.5). Then, for any $\delta \in (0, 1/6)$, there exists $u_{0,\delta}$ such that $u_{0,\delta} \in C^{0,a}(\overline{\Omega})$ for any $a \in (0, 1/2)$. Moreover,*

$$-1 + 3\delta \leq u_{0,\delta}(x) \leq 1 - 3\delta \quad \forall x \in \overline{\Omega}. \quad (3.1)$$

Finally, we have that

$$u_{0,\delta} \rightarrow u_0 \quad \text{weakly in } V, \quad (3.2)$$

$$\mathcal{E}(u_{0,\delta}) \leq c(1 + \mathcal{E}(u_0)) \quad \forall \delta \in (0, 1/6). \quad (3.3)$$

PROOF. First of all, we set

$$u_{0,\delta}^{(1)} := \min \{1 - 3\delta, \max\{u_0, -1 + 3\delta\}\}, \quad (3.4)$$

so that $u_{0,\delta}^{(1)}$ satisfies a.e. in Ω the equivalent of (3.1). Moreover, it is clear that $\mathcal{E}(u_{0,\delta}^{(1)}) \leq \mathcal{E}(u_0)$ for all δ . Next, we define $z_{0,\delta}^{(1)} := \phi(u_{0,\delta}^{(1)}) = \arcsin u_{0,\delta}^{(1)}$. Then, we proceed by singular perturbation, defining $z_{0,\delta}$ as the unique solution of the elliptic problem

$$z_{0,\delta} + \delta A z_{0,\delta} = z_{0,\delta}^{(1)}. \quad (3.5)$$

Being $z_{0,\delta}^{(1)} \in V$, then, by elliptic regularity (recall that Ω is a smooth domain), $z_{0,\delta} \in H^3(\Omega)$ for all δ . Setting $u_{0,\delta} := \sin z_{0,\delta}$, a direct check permits to verify that (at least) $u_{0,\delta} \in H^2(\Omega)$ for all δ . Hence, $u_{0,\delta}$ is Hölder continuous, as desired. Moreover, by monotonicity of ϕ and a standard maximum principle argument, it is clear that (3.1) holds.

The key step consists in proving (3.3). Actually, it is obvious that $\|F(u_{0,\delta})\|_{L^1(\Omega)} \leq c$ (cf. (2.4)). To control the gradient term of \mathcal{E} , we test (3.5) by $A z_{0,\delta}$. We obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega} a(u_{0,\delta}) |\nabla u_{0,\delta}|^2 &= \|\nabla z_{0,\delta}\|^2 \leq \|\nabla z_{0,\delta}^{(1)}\|^2 \\ &= \frac{1}{2} \int_{\Omega} a(u_{0,\delta}^{(1)}) |\nabla u_{0,\delta}^{(1)}|^2 \leq \frac{1}{2} \int_{\Omega} a(u_0) |\nabla u_0|^2 \leq \mathcal{E}(u_0) + c, \end{aligned} \quad (3.6)$$

as desired. Hence, we have (3.3). To conclude, we have to prove (3.2), which is however an immediate consequence of standard weak compactness arguments. The proof is complete. \blacksquare

As a next step, we also provide a modification of the function f given by (1.6). Namely, for all $\delta \in (0, 1/6)$, we take $f_{\delta} \in C^1(-1 + \delta, 1 - \delta) \rightarrow \mathbb{R}$ such that f_{δ} is monotone, $|f_{\delta}(r)| \geq |f(r)|$ for all $r \in (-1 + \delta, 1 - \delta)$, and

$$f_{\delta}(r) = f(r) \quad \text{if } r \in [-1 + 2\delta, 1 - 2\delta], \quad (3.7)$$

$$\lim_{|r| \rightarrow 1 - \delta} f_{\delta}(r) \operatorname{sign} r = +\infty. \quad (3.8)$$

It is obvious that, for any $\delta \in (0, 1/6)$, a function f_{δ} with the above properties exists.

Finally, we modify a by taking $a_{\delta} \in C^2(\mathbb{R}; \mathbb{R})$ such that

$$a_{\delta}(r) = a(r) \quad \forall r \in [-1 + \delta, 1 - \delta]. \quad (3.9)$$

In particular,

$$a_{\delta}''(r) \geq 0, \quad \left(\frac{1}{a_{\delta}}\right)''(r) = -1 \quad \forall r \in [-1 + \delta, 1 - \delta]. \quad (3.10)$$

Outside $(-1, 1)$, a_{δ} is taken as a constant $K_{\delta} > 0$ (exploding as $\delta \searrow 0$), whereas for $|r| \in (1 - \delta, 1)$, a_{δ} is chosen in such a way to have

$$1 \leq a_{\delta}(r) \leq K_{\delta} \quad \forall r \in \mathbb{R}. \quad (3.11)$$

Then, for $\delta \in (0, 1/6)$, we can consider the system

$$u_{\delta,t} - \Delta w_\delta = 0, \quad \text{in } (0, T) \times \Omega, \quad (3.12)$$

$$w_\delta = -a_\delta(u_\delta)\Delta u_\delta - \frac{a'_\delta(u_\delta)}{2}|\nabla u_\delta|^2 + f_\delta(u_\delta) - \lambda u_\delta + \varepsilon u_{\delta,t}, \quad \text{in } (0, T) \times \Omega, \quad (3.13)$$

$$u_\delta|_{t=0} = u_{0,\delta}, \quad \text{in } \Omega, \quad (3.14)$$

$$\partial_{\mathbf{n}} u_\delta = \partial_{\mathbf{n}} w_\delta = 0, \quad \text{on } \Gamma. \quad (3.15)$$

We then have:

Theorem 3.2. *Let f and a be given respectively by (1.6), (1.7), and let u_0 satisfy (2.3) and (2.5). For $\delta \in (0, 1/6)$, let $u_{0,\delta}$ be defined by Lemma 3.1 and f_δ , a_δ be given by (3.7)-(3.11). Then, there exists one and only one solution (u_δ, w_δ) to system (3.12)-(3.15) with the regularity*

$$u_\delta \in W^{1,\infty}(0, T; V') \cap H^1(0, T; V) \cap L^\infty(0, T; H^2(\Omega)), \quad \varepsilon u_\delta \in W^{1,\infty}(0, T; H), \quad (3.16)$$

$$w_\delta \in L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega)), \quad \varepsilon w_\delta \in L^\infty(0, T; H^2(\Omega)). \quad (3.17)$$

PROOF. We claim that this result is essentially a consequence of the results of [26]. Actually, we see that, for any $\delta \in (0, 1/6)$, a_δ satisfies the assumptions [26, (2.1)-(2.2)] (where -1 and 1 replace -2 and 2 in [26, (2.2)]) and [26, (6.1)] (where $[-1, 1]$ is replaced by $[-1 + \delta, 1 - \delta]$, cf. (3.10)).

Moreover, f_δ satisfies [26, (2.3)-(2.4)], with $(-1 + \delta, 1 - \delta)$ replacing $(-1, 1)$. Thus, we can apply [26, Theorems 5.1, 6.1, 6.2] which give the existence and uniqueness of a weak solution (u_δ, w_δ) to (3.12)-(3.15). The regularity of this solution is specified by [26, (4.3)-(4.4) and (6.14)]. More precisely, since the initial datum $u_{0,\delta}$ is smooth and separated in the uniform norm from the singular values $\pm(1 - \delta)$ of f_δ due to (3.1), we have here that [26, (6.14)] holds starting from the initial time, i.e., with $\tau = 0$. Moreover, a closer inspection of [26, Proof of Theorem 6.2] (see in particular estimates (6.19)-(6.20) therein) permits to see that the additional regularity for $u_{\delta,t}$ stated in (3.16) holds over $(0, T)$. The $L^2(0, T; H^3(\Omega))$ regularity of w_δ follows from the $L^2(0, T; V)$ regularity of $u_{\delta,t}$ and elliptic regularity estimates applied to (3.12).

Hence, collecting all the information coming from the results of [26], we obtain exactly (3.16)-(3.17). This concludes the proof. \blacksquare

Notice now that, as a consequence of (3.16)-(3.17) and of the arguments in [26], we have, more precisely, the a-priori estimate

$$\|u_\delta\|_{H^1(0,T;V')} + \|u_\delta\|_{L^\infty(0,T;H^2(\Omega))} \leq Q(\|u_{0,\delta}\|_{H^2(\Omega)}, \delta^{-1}). \quad (3.18)$$

By interpolation and embedding properties of Sobolev spaces, we then obtain

$$\|u_\delta\|_{C^{0,b}([0,T] \times \overline{\Omega})} \leq Q(\|u_{0,\delta}\|_{H^2(\Omega)}, \delta^{-1}), \quad \text{for some } b > 0. \quad (3.19)$$

Thus, u_δ is globally Hölder continuous. Since $u_{0,\delta}$ satisfies (3.1), we can deduce that, once the initial datum u_0 is assigned, then for all $\delta \in (0, 1/6)$ there exists a (computable) time $T_\delta > 0$ depending on δ and u_0 such that

$$-1 + 2\delta \leq u_\delta(t, x) \leq 1 - 2\delta \quad \text{for all } (t, x) \in [0, T_\delta] \times \overline{\Omega}. \quad (3.20)$$

However, over $[-1 + 2\delta, 1 - 2\delta]$, a_δ coincides with a by (3.9) and f_δ coincides with f by (3.7). Hence, we have obtained the

Corollary 3.3. *Let f and a be given by (1.6), (1.7), and let u_0 satisfy (2.3) and (2.5). Let $\delta \in (0, 1/6)$ and let $u_{0,\delta}$ be given by Lemma 3.1. Then, there exist a time $T_\delta > 0$ depending on u_0 and δ , and a “classical” solution (u_δ, w_δ) to Problem (P), with initial datum $u_{0,\delta}$, over the time interval $(0, T_\delta)$.*

4 A priori estimates

In this section, we derive a number of a-priori estimates for the solutions of our system. We stress that the procedure leading to these estimates can be rigorously justified at least for “classical” solutions to Problem (P) in the sense of Definition 2.1. Indeed, owing to Remark 2.2, these solutions can be thought to be as smooth as we need (possibly paying the price of additionally regularizing u_0).

In particular, the estimates proved below will hold for local strong solutions given by Theorem 3.2. On the other hand, for weaker notions of solutions the procedure below may just have a formal character due to insufficient regularity of test functions. We will clarify this point in Section 5 below. Here, we will proceed assuming that everything is regular enough for our purposes. Moreover, since the estimates we are going to derive will have a global-in time character, with some abuse of notation we will directly work on the time interval $[0, T]$. The underlying extension argument will be also detailed in the next section.

We can now start detailing our estimates.

Energy estimate. We test (1.1) by w and (1.2) by u_t . This gives rise to the energy equality (1.18), whence we obtain the estimate

$$\|u_t\|_{L^2(0,T;V')}^2 + \varepsilon \|u_t\|_{L^2(0,T;H)}^2 + \|\nabla w\|_{L^2(0,T;H)}^2 \leq Q(\mathbb{E}_0), \quad (4.1)$$

$$\|F(u)\|_{L^\infty(0,T;L^1(\Omega))} \leq Q(\mathbb{E}_0). \quad (4.2)$$

Thus, by definition (1.5) of F , we infer in particular that

$$-1 \leq u \leq 1 \quad \text{a.e. in } (0, T) \times \Omega. \quad (4.3)$$

Moreover, recalling also (2.12), we obtain

$$\|u\|_{L^\infty(0,T;V)} + \|z\|_{L^\infty(0,T;V)} \leq Q(\mathbb{E}_0). \quad (4.4)$$

Estimate on time derivatives. Following the lines of [26, Theorem 6.1], we indicate by J the gradient part of the energy, i.e.,

$$J : V \rightarrow [0, +\infty), \quad J(u) := \int_{\Omega} \frac{a(u)}{2} |\nabla u|^2. \quad (4.5)$$

Then, we can (formally) compute the first derivative of J , given by

$$\langle J'(u), v \rangle = \int_{\Omega} \left(a(u) \nabla u \cdot \nabla v + \frac{a'(u)}{2} |\nabla u|^2 v \right), \quad (4.6)$$

as well as the second derivative

$$\langle J''(u)v, z \rangle = \int_{\Omega} \left(\frac{a''(u)|\nabla u|^2 v z}{2} + a'(u)v \nabla u \cdot \nabla z + a'(u)z \nabla u \cdot \nabla v + a(u) \nabla v \cdot \nabla z \right). \quad (4.7)$$

To be more precise, if u is a smooth solution (and in particular it is separated from singularities, i.e., it satisfies (2.23)), formulas (4.6) and (4.7) make sense for $u = u(t)$ at any time $t \in [0, T]$ and, indeed, we have that $J'(u) \in V'$ and $J''(u) \in \mathcal{L}(V, V')$.

From (4.7), we then obtain in particular

$$\begin{aligned} \langle J''(u)v, v \rangle &= \int_{\Omega} \left(\frac{a''(u)|\nabla u|^2 v^2}{2} + 2a'(u)v \nabla u \cdot \nabla v + a(u)|\nabla v|^2 \right) \\ &\geq \int_{\Omega} \left(a(u) - \frac{2a'(u)^2}{a''(u)} \right) |\nabla v|^2. \end{aligned} \quad (4.8)$$

We can now test (1.1) by tu_t and add the time derivative of (1.2) tested by tu_t . This leads to

$$\frac{t}{2} \frac{d}{dt} \|\nabla w\|^2 + \frac{t\varepsilon}{2} \frac{d}{dt} \|u_t\|^2 + t \langle J''(u)u_t, u_t \rangle + t \int_{\Omega} f'(u)u_t^2 = \lambda t \|u_t\|^2. \quad (4.9)$$

Hence, using (4.8), noting that

$$a(u) - \frac{2a'(u)^2}{a''(u)} = \frac{2}{1+3u^2}, \quad (4.10)$$

and using (4.3), we arrive at the inequality

$$\begin{aligned} & \frac{d}{dt} \left(\frac{t}{2} \|\nabla w\|^2 + \frac{\varepsilon t}{2} \|u_t\|^2 \right) + t \int_{\Omega} \left(\frac{2}{1+3u^2} |\nabla u_t|^2 \right) + t \int_{\Omega} f'(u) u_t^2 \\ & \leq \lambda t \|u_t\|^2 + \frac{1}{2} \|\nabla w\|^2 + \frac{\varepsilon}{2} \|u_t\|^2 \leq \frac{t}{4} \|\nabla u_t\|^2 + c(1+t) \|\nabla w\|^2 + \frac{\varepsilon}{2} \|u_t\|^2. \end{aligned} \quad (4.11)$$

Recalling (4.1) and noting that $2/(1+3u^2) \geq 1/2$ since u takes values in $[-1, 1]$, integration in time of (4.11) gives, for any $\tau \in (0, T)$,

$$\|u_t\|_{L^\infty(\tau, T; V')} + \|\nabla w\|_{L^\infty(\tau, T; H)} \leq Q(\mathbb{E}_0, \tau^{-1}), \quad (4.12)$$

$$\varepsilon \|u_t\|_{L^\infty(\tau, T; H)} + \|u_t\|_{L^2(\tau, T; V)} \leq Q(\mathbb{E}_0, \tau^{-1}). \quad (4.13)$$

Estimate of $f(u)$. We test (1.2) by $u - m$. Integrating by parts the terms depending on a , we obtain

$$\int_{\Omega} \left(a(u) + \frac{a'(u)}{2} (u - m) \right) |\nabla u|^2 + \int_{\Omega} f(u) (u - m) = (w + \lambda u - \varepsilon u_t, u - m). \quad (4.14)$$

We have to estimate some terms. Firstly, proceeding as in [19, Appendix], it is not difficult to prove that

$$\int_{\Omega} f(u) (u - m) \geq \frac{1}{2} \|f(u)\|_{L^1(\Omega)} - c_m. \quad (4.15)$$

We notice that assumption (2.5) is used here.

We now observe that

$$\int_{\Omega} \left(a(u) + \frac{a'(u)}{2} (u - m) \right) |\nabla u|^2 = \int_{\Omega} \left(\frac{2u(u - m)}{(1 - u^2)^2} + \frac{2}{1 - u^2} \right) |\nabla u|^2. \quad (4.16)$$

Moreover, there exist constants $\kappa_m > 0$, $c_m \geq 0$ such that

$$\frac{2u(u - m)}{(1 - u^2)^2} \geq \kappa_m \frac{1}{(1 - u^2)^2} - c_m \geq \kappa_m \frac{u^2}{(1 - u^2)^2} - c_m. \quad (4.17)$$

Thus, recalling (1.7), we get

$$\int_{\Omega} \left(a(u) + \frac{a'(u)}{2} (u - m) \right) |\nabla u|^2 \geq \kappa_m \|\nabla f(u)\|^2 - c_m \|\nabla u\|^2. \quad (4.18)$$

Finally, noting that u takes values in $[-1, 1]$, using estimate (4.4), and applying the Poincaré-Wirtinger inequality, we have

$$(w + \lambda u - \varepsilon u_t, u - m) = (w - w_{\Omega} + \lambda u - \varepsilon u_t, u - m) \leq c(\|\nabla w\| + 1 + \varepsilon \|u_t\|). \quad (4.19)$$

Thus, collecting the above considerations, (4.14) gives

$$\|\nabla f(u)\|^2 + \|f(u)\|_{L^1(\Omega)} \leq c(\|\nabla u\|^2 + \|\nabla w\| + 1 + \varepsilon \|u_t\|). \quad (4.20)$$

Squaring (4.20), using (4.15)-(4.19), and integrating in time, we arrive at

$$\|\nabla f(u)\|_{L^4(0, T; H)} + \|f(u)\|_{L^2(0, T; L^1(\Omega))} \leq Q(\mathbb{E}_0), \quad (4.21)$$

whence, more precisely,

$$\|f(u)\|_{L^2(0, T; V)} \leq Q(\mathbb{E}_0). \quad (4.22)$$

Taking instead the essential supremum of (4.20) as t ranges in (τ, T) , and using (4.12)-(4.13), a straightforward modification of the above procedure leads to

$$\|f(u)\|_{L^\infty(\tau, T; V)} \leq Q(\mathbb{E}_0, \tau^{-1}) \quad \forall \tau \in (0, T). \quad (4.23)$$

Next, integrating (1.2) in space, using (2.7), (4.3), (1.7), and noting that $(u_t)_\Omega \equiv 0$, we get

$$\begin{aligned} |w_\Omega| &\leq c \left(\int_\Omega \frac{a'(u)}{2} |\nabla u|^2 + \|f(u)\|_{L^1(\Omega)} + 1 \right) \\ &\leq c \left(\|\nabla f(u)\|^2 + \|f(u)\|_{L^1(\Omega)} + 1 \right). \end{aligned} \quad (4.24)$$

Thus, squaring, integrating in time, using (4.21), and recalling the last (4.1), we infer

$$\|w\|_{L^2(0,T;V)} \leq Q(\mathbb{E}_0). \quad (4.25)$$

Taking the essential supremum in (4.24) as $t \in (\tau, T)$, and recalling (4.12) and (4.23), we also get

$$\|w\|_{L^\infty(\tau,T;V)} \leq Q(\mathbb{E}_0, \tau^{-1}) \quad \forall \tau \in (0, T). \quad (4.26)$$

Estimate of z . We consider the equivalent formulation (2.13) and test it by $-\Delta z$. This gives

$$\int_\Omega 2\phi'(u)|\Delta z|^2 + (f'(u)\nabla u, \nabla z) = (w + \lambda u - \varepsilon u_t, -\Delta z). \quad (4.27)$$

Thus, using the monotonicity of f and ϕ , noting that $\phi'(u) \geq 1/2$ for all $u \in (-1, 1)$, and that $(f'(u)\nabla u, \nabla z) \geq 2\|\nabla z\|^2$, we can control the right hand side this way:

$$(w + \lambda u - \varepsilon u_t, -\Delta z) \leq \frac{1}{2}\|\Delta z\|^2 + c(\|w\|^2 + 1 + \varepsilon^2\|u_t\|^2). \quad (4.28)$$

Hence, integrating (4.27) in time and recalling (4.1), (4.4) and (4.25), we arrive at

$$\|z\|_{L^2(0,T;H^2(\Omega))} \leq Q(\mathbb{E}_0). \quad (4.29)$$

Taking instead the (essential) supremum of (4.27) as t ranges in (τ, T) for $\tau > 0$ and using (4.12)-(4.13) and (4.26), we obtain

$$\|z\|_{L^\infty(\tau,T;H^2(\Omega))} \leq Q(\mathbb{E}_0, \tau^{-1}). \quad (4.30)$$

The above relations permit to improve also the bounds on u . Actually, computing directly the Laplacean of $u = \sin z$ and using (4.3) together with the Gagliardo-Nirenberg inequality (cf., e.g., [22, Theorem p. 125])

$$\|\nabla y\|_{L^4(\Omega)} \leq c_\Omega \|y\|_{H^2(\Omega)}^{1/2} \|y\|_{L^\infty(\Omega)}^{1/2} + \|y\| \quad \forall y \in H^2(\Omega), \quad (4.31)$$

it is not difficult to arrive at

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq Q(\mathbb{E}_0), \quad (4.32)$$

$$\|u\|_{L^\infty(\tau,T;H^2(\Omega))} \leq Q(\mathbb{E}_0, \tau^{-1}). \quad (4.33)$$

First entropy estimate and separation property. The estimates obtained up to this moment yield a control of the functions z and u up to their second space derivatives (cf. (4.29)-(4.30) and (4.32)-(4.33)), and of the nonlinear term $f(u)$ up to its first space derivatives (cf. (4.22) and (4.23)). However, this still seems not sufficient to pass to the limit in the equation (1.2), even if its equivalent formulation (2.13) is considered. Indeed, from (4.22) and (4.23) we get a control of the term $f(u)$, that explodes logarithmically fast as $|u| \nearrow 1$. On the other hand, even in formulation (2.13) one faces the term $\phi'(u)$ which is much more singular since it explodes as a negative power of $(1 - u^2)$. To control it in some L^p -norm we need more refined estimates of the so-called *entropy* type and, in particular, we need to refer to the formulation (2.17) (we recall that all formulations are equivalent, at least for sufficiently smooth solutions). Usage of this technique requires the convexity assumption on Ω asked in the statement of Theorem 2.4.

The basic tool we need consists in an integration by parts formula due to Dal Passo, Garcke and Grün ([8, Lemma 2.3]):

Lemma 4.1. *Let $h \in W^{2,\infty}(\mathbb{R})$ and $z \in W$. Then,*

$$\begin{aligned} \int_{\Omega} h'(z) |\nabla z|^2 \Delta z &= -\frac{1}{3} \int_{\Omega} h''(z) |\nabla z|^4 \\ &+ \frac{2}{3} \int_{\Omega} h(z) (|D^2 z|^2 - |\Delta z|^2) + \frac{2}{3} \int_{\Gamma} h(z) II(\nabla z), \end{aligned} \quad (4.34)$$

where $II(\cdot)$ denotes the second fundamental form of Γ .

Then, we test (2.17) by

$$-\Delta \widehat{m}(v) = -\operatorname{div}(m(v) \nabla v) = -m(v) \Delta v - m'(v) |\nabla v|^2, \quad (4.35)$$

with the function m to be chosen later. This gives

$$\begin{aligned} \int_{\Omega} m(v) |\nabla v|^2 + \int_{\Omega} \left(m(v) |\Delta v|^2 - \frac{m'(v) j(v)}{2} |\nabla v|^4 + \left(m'(v) - \frac{m(v) j'(v)}{2} \right) \Delta v |\nabla v|^2 \right) \\ = (m(v) \nabla v, \nabla w + \lambda \nabla u) + (\varepsilon u_t, m(v) \Delta v + m'(v) |\nabla v|^2). \end{aligned} \quad (4.36)$$

Now, applying Lemma 4.1 to the last integral on the left hand side of (4.36) we infer

$$\begin{aligned} \int_{\Omega} \left(m'(v) - \frac{m(v) j(v)}{2} \right) \Delta v |\nabla v|^2 &= -\frac{1}{3} \int_{\Omega} \left(m''(v) - \frac{m'(v) j(v)}{2} - \frac{m(v) j'(v)}{2} \right) |\nabla v|^4 \\ &+ \frac{2}{3} \int_{\Omega} \left(m(v) - \frac{\widehat{m} j(v)}{2} + K \right) (|D^2 v|^2 - |\Delta v|^2) + \frac{2}{3} \int_{\Gamma} \left(m(v) - \frac{\widehat{m} j(v)}{2} + K \right) II(\nabla v), \end{aligned} \quad (4.37)$$

where $K > 0$ is an integration constant that will be chosen later on, and the notation (2.9) is used.

Substituting (4.37) into (4.36), we get on the left hand side the following “hopefully good” terms

$$\int_{\Omega} m(v) |\nabla v|^2 + \int_{\Omega} m(v) |\Delta v|^2 + \frac{1}{3} \int_{\Omega} \left(-m''(v) - m'(v) j(v) + \frac{m(v) j'(v)}{2} \right) |\nabla v|^4, \quad (4.38)$$

where we notice that

$$j'(v) = \frac{2e^v}{(e^v + 1)^2}. \quad (4.39)$$

We can now specify our choice of m as

$$m(v) = \frac{1}{2(1 + v^2)}. \quad (4.40)$$

Actually, this expression arises since we need m to decay not too fast at infinity (otherwise we do not get enough information from it), but at the same time we need it to be summable (cf. (4.47) below). The above choice gives

$$\widehat{m}(v) = \frac{1}{2} \arctan v, \quad (4.41)$$

as well as

$$m'(v) = -\frac{v}{(1 + v^2)^2}, \quad m''(v) = \frac{3v^2 - 1}{(1 + v^2)^3}. \quad (4.42)$$

Now, noting that $m j' \geq 0$, we can easily observe that

$$-m''(v) - m'(v) j(v) + \frac{m(v) j'(v)}{2} \geq \frac{(v + v^3) j(v) + 1 - 3v^2}{(1 + v^2)^3}. \quad (4.43)$$

Now, recalling (2.14), for a suitable $M > 0$ (e.g., we can take $M = 12$ here), a direct computation shows that

$$\frac{(v + v^3) j(v) + 1 - 3v^2}{(1 + v^2)^3} \geq \frac{1}{4} \frac{1 + |v|^3}{(1 + v^2)^3} \quad \forall |v| \geq M. \quad (4.44)$$

On the other hand,

$$\frac{(v + v^3)j(v) + 1 - 3v^2}{(1 + v^2)^3} \geq \frac{1}{(1 + v^2)^3} - \frac{3v^2}{(1 + v^2)^3} \geq \kappa_M \frac{1 + |v|^3}{(1 + v^2)^3} - c_M \frac{|v|}{(1 + v^2)^3} \quad \forall |v| \leq M. \quad (4.45)$$

Summarizing, we have

$$-m''(v) - m'(v)j(v) + \frac{m(v)j'(v)}{2} \geq \kappa \frac{1 + |v|^3}{(1 + v^2)^3} - c \frac{|v|}{(1 + v^2)^3} \chi_{\{|v| \leq M\}}, \quad (4.46)$$

where χ denotes the characteristic function.

Now, we can choose K so large that the function

$$\left(m(v) - \frac{\widehat{mj}(v)}{2} + K\right) \quad (4.47)$$

is strictly positive (and bounded, of course). Then, thanks the *convexity* assumption on Ω (that entails positive definiteness of the second fundamental form), the latter two terms in (4.37) are positive.

Collecting these observations, we can deduce from (4.36) the estimate

$$\int_{\Omega} m(v)|\nabla v|^2 + \int_{\Omega} m(v)|\Delta v|^2 + \kappa \int_{\Omega} \frac{1 + |v|^3}{(1 + v^2)^3} |\nabla v|^4 \leq \sum_{i=1}^4 T_i, \quad (4.48)$$

and we have to control the “bad” terms on the right hand side:

$$T_1 := c \int_{\{|v| \leq M\}} \frac{|v|}{(1 + v^2)^3} |\nabla v|^4, \quad (4.49)$$

$$T_2 := \int_{\Omega} m(v) \nabla v \cdot \nabla w, \quad (4.50)$$

$$T_3 := \lambda \int_{\Omega} m(v) \nabla v \cdot \nabla u, \quad (4.51)$$

$$T_4 := (\varepsilon u_t, m(v) \Delta v + m'(v) |\nabla v|^2). \quad (4.52)$$

To do this, we first notice that

$$|T_2 + T_3| \leq \frac{1}{4} \int_{\Omega} m(v) |\nabla v|^2 + c(\|\nabla w\|^2 + \|\nabla u\|^2). \quad (4.53)$$

Next, we have to control T_1 . Then, we can note that there exists $\delta \in (0, 1)$, depending only on M , such that the restriction $f : [-1 + \delta, 1 - \delta] \rightarrow [-M, M]$ is bijective and Lipschitz continuous together with its inverse j . Thus, using (4.3) and inequality (4.31), we deduce

$$\begin{aligned} |T_1| &\leq c \int_{\{|v| \leq M\}} |\nabla v|^4 \leq c \int_{\{|v| \leq M\}} f'(u)^4 |\nabla u|^4 \\ &\leq c \int_{\{|u| \leq 1 - \delta\}} \frac{1}{(1 - u^2)^4} |\nabla u|^4 \leq c_{\delta} \int_{\{|u| \leq 1 - \delta\}} |\nabla u|^4 \\ &\leq c_{\delta} \int_{\Omega} |\nabla u|^4 \leq c_{\delta} \|u\|_{L^{\infty}(\Omega)}^2 (1 + \|u\|_{H^2(\Omega)}^2) \leq c_{\delta} (1 + \|u\|_{H^2(\Omega)}^2). \end{aligned} \quad (4.54)$$

Finally, we have to control T_4 . We have

$$|T_4| \leq \frac{1}{4} \int_{\Omega} m(v) |\Delta v|^2 + c_{\sigma} \varepsilon^2 \|u_t\|^2 + \sigma \int_{\Omega} m'(v)^2 |\nabla v|^4, \quad (4.55)$$

for small $\sigma > 0$ to be chosen later. Then, we can go on as follows:

$$\sigma \int_{\Omega} m'(v)^2 |\nabla v|^4 = \sigma \int_{\Omega} \frac{v^2}{(1 + v^2)^4} |\nabla v|^4 \leq \sigma \int_{\Omega} \frac{1 + |v|^3}{(1 + v^2)^3} |\nabla v|^4. \quad (4.56)$$

Hence, for σ sufficiently small, the last integral is controlled by the last term on the left hand side of (4.48). On the other hand, the first term on the right hand side of (4.55) is controlled by the corresponding one on the left hand side of (4.48).

Now, a direct computation permits to see that

$$\int_{\Omega} \frac{1+|v|^3}{(1+v^2)^3} |\nabla v|^4 \geq \int_{\Omega} \frac{|v|^3}{(1+v^2)^3} \frac{|v|}{(1+v^2)^{1/2}} |\nabla v|^4 \geq \kappa \int_{\Omega} |\nabla(1+v^2)^{1/8}|^4. \quad (4.57)$$

Thus, collecting (4.53)-(4.57), (4.48) gives

$$\begin{aligned} & \int_{\Omega} m(v) |\nabla v|^2 + \int_{\Omega} m(v) |\Delta v|^2 + \int_{\Omega} |\nabla(1+v^2)^{1/8}|^4 \\ & \leq c(\|\nabla w\|^2 + \|\nabla u\|^2 + 1 + \|u\|_{H^2(\Omega)}^2 + \varepsilon^2 \|u_t\|^2). \end{aligned} \quad (4.58)$$

Hence, integrating in time, and using (4.1), (4.4) and (4.32), we obtain

$$\|m^{1/2}(v) \Delta v\|_{L^2(0,T;H)} + |(1+v^2)^{1/8}|_{L^4(0,T;W^{1,4}(\Omega))} \leq Q(\mathbb{E}_0). \quad (4.59)$$

On the other hand, taking the essential supremum in (4.58) as t ranges in (τ, T) for $\tau > 0$, and using (4.12), (4.13) and (4.33), we arrive at

$$\|m^{1/2}(v) \Delta v\|_{L^\infty(\tau,T;H)} + |(1+v^2)^{1/8}|_{L^\infty(\tau,T;W^{1,4}(\Omega))} \leq Q(\mathbb{E}_0, \tau^{-1}). \quad (4.60)$$

By the continuous embedding $W^{1,4}(\Omega) \subset L^\infty(\Omega)$ we have in particular

$$|(1+v^2)^{1/8}|_{L^\infty((\tau,T) \times \Omega)} \leq Q(\mathbb{E}_0, \tau^{-1}). \quad (4.61)$$

In terms of u the above estimate gives rise to the separation property

$$-1 + \epsilon \leq u(x, t) \leq 1 - \epsilon \quad \text{a.e. in } (\tau, T) \times \Omega, \quad (4.62)$$

for all $\tau > 0$, with $\epsilon > 0$ depending on τ .

Refined entropy estimate. We repeat the entropy estimate of before taking now, in place of (4.40),

$$m(v) = \frac{1}{2(1+v^2)^p}, \quad (4.63)$$

where the choice of $p \in (1/2, 1]$ will be made precise later on. Then, we have

$$m'(v) = -\frac{pv}{(1+v^2)^{p+1}}, \quad m''(v) = \frac{(2p^2 + p)v^2 - p}{(1+v^2)^{p+2}}. \quad (4.64)$$

Thus, a straightforward modification of (4.43)-(4.45) leads to

$$-m''(v) - m'(v)j(v) + \frac{m(v)j'(v)}{2} \geq \kappa \frac{1+|v|^3}{(1+v^2)^{p+2}} - c \frac{|v|}{(1+v^2)^{p+2}} \chi_{\{|v| \leq M\}}, \quad (4.65)$$

whereas the equivalent of (4.57) gives now rise to

$$\int_{\Omega} \frac{1+|v|^3}{(1+v^2)^{p+2}} |\nabla v|^4 \geq \kappa \int_{\Omega} |\nabla(1+v^2)^{\frac{3}{8}-\frac{p}{4}}|^4. \quad (4.66)$$

Now, let us observe that, since $p > 1/2$, then the function \widehat{mj} is bounded. Thus, we can still take K so large, depending of course on p , that the function in (4.47) is strictly positive. Moreover, the terms corresponding to T_j , $j = 1, \dots, 4$, can be controlled similarly as before. Thus, integrating in time the p -analogue of (4.48), we obtain

$$\int_0^T \int_{\Omega} |\nabla(1+v^2)^{\frac{3}{8}-\frac{p}{4}}|^4 \leq Q(\mathbb{E}_0). \quad (4.67)$$

Let us now test (2.17) by v . This gives

$$\|\nabla v\|^2 + \|v\|^2 + \int_{\Omega} \frac{j(v)v}{2} |\nabla v|^2 \leq (w + \lambda u - \varepsilon u_t, v). \quad (4.68)$$

Then, integrating in time, noting that $1 + j(v)v \geq \kappa(1 + v^2)^{1/2}$, recalling (4.1) and using Hölder's and Young's inequalities to estimate the right hand side, we arrive at

$$\int_0^T \int_{\Omega} (1 + v^2)^{1/2} |\nabla v|^2 \leq Q(\mathbb{E}_0). \quad (4.69)$$

Then, let us define

$$\Omega^+(t) := \{x \in \Omega : |v(x, t)| \geq 1\}, \quad \Omega^-(t) := \{x \in \Omega : |v(x, t)| \leq 1\}. \quad (4.70)$$

Let us now see that (4.67) and (4.69) permit to prove higher integrability properties of ∇v . Firstly, recalling (4.4) and (4.32), and using inequality (4.31), we obtain

$$\|u\|_{L^4(0, T; W^{1,4}(\Omega))} \leq Q(\mathbb{E}_0). \quad (4.71)$$

Hence, noting that j is locally Lipschitz continuous with its inverse, we get an analogous information for v in the space-time set where it is small:

$$\int_0^T \int_{\Omega^-(t)} |\nabla v|^4 \leq Q(\mathbb{E}_0). \quad (4.72)$$

On the other hand, in $\Omega^+(t)$ we can write, for $\eta > 0$,

$$\begin{aligned} |\nabla v|^2 &= (v(1 + v^2)^{-\eta} \nabla v) \cdot \left(\frac{(1 + v^2)^{\eta}}{v} \nabla v \right) \\ &= c_{\eta} \nabla(1 + v^2)^{1-\eta} \cdot \left(\frac{(1 + v^2)^{\eta}}{v} \nabla v \right). \end{aligned} \quad (4.73)$$

Thus, choosing $\eta = \frac{5}{8} + \frac{p}{4}$, we have $1 - \eta = \frac{3}{8} - \frac{p}{4}$. Then, we can take $p = \frac{1}{2} + \epsilon$, with $\epsilon > 0$ as small as we want. We then obtain that $\eta - \frac{1}{4} = \frac{1}{2} + \frac{\epsilon}{4}$. Thus, recalling (4.67) and (4.69), we arrive at

$$|\nabla v|^2 \leq c_{\eta} \left| \underbrace{\nabla(1 + v^2)^{1-\eta}}_{L^4} \cdot \underbrace{((1 + v^2)^{1/4} \nabla v)}_{L^2} \underbrace{\frac{(1 + v^2)^{\eta-1/4}}{v}}_{L^r} \right|, \quad (4.74)$$

where we can take r strictly greater than 4 since for $|v| \geq 1$ we have

$$\frac{(1 + v^2)^{\eta-1/4}}{|v|} = \frac{(1 + v^2)^{\frac{1}{2} + \frac{\epsilon}{4}}}{|v|} \sim |v|^{\frac{\epsilon}{2}}, \quad (4.75)$$

and we know from (4.22) that v is controlled in $L^2(0, T; H)$. Thus, estimating the right hand side of (4.74) by Young's inequality, integrating first over $\Omega^+(t)$ and then for $t \in (0, T)$, we obtain that

$$\int_0^T \int_{\Omega^+(t)} |\nabla v|^q \leq Q(\mathbb{E}_0), \quad \text{for some } q \text{ **strictly** larger than } 2, \quad (4.76)$$

and, of course, combining with (4.72),

$$\int_0^T \int_{\Omega} |\nabla v|^q \leq Q(\mathbb{E}_0), \quad \text{for some } q \text{ **strictly** larger than } 2. \quad (4.77)$$

5 Existence and uniqueness of weak solutions

We detail here the proof of Theorem 2.4, which is largely based on the estimates derived in the previous section. As a first step, however, we show uniqueness, which works similarly to [26]. Indeed, the key assumption [26, (6.1)] is satisfied by our function a (we have, indeed, that $(1/a)'' \equiv -1$).

Then, let us take a couple of weak solutions (u_1, w_1) and (u_2, w_2) originating from the same initial datum u_0 , and assume that both are “classical” (and in particular satisfy the separation property (2.23)), for strictly positive times.

Setting $(u, w) := (u_1, w_1) - (u_2, w_2)$, we can write both (2.28) and (2.29) for the two solutions and take the difference. Using notation (4.5) we get

$$u_t + Aw = 0, \quad (5.1)$$

$$w = J'(u_1) - J'(u_2) + f(u_1) - f(u_2) - \lambda u + \varepsilon u_t. \quad (5.2)$$

Then, we can test (5.1) by $A^{-1}u$, (5.2) by u , and take the difference. Actually, the operator A is invertible as it is restricted to 0-mean valued functions (as in the case of u due to conservation of mass). Noting that

$$(Aw, A^{-1}u) = (A(w - w_\Omega), A^{-1}u) = (w - w_\Omega, u) = (w, u), \quad (5.3)$$

we then obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{V'}^2 + \varepsilon \|u\|^2 \right) + \langle J'(u_1) - J'(u_2), u \rangle + (f(u_1) - f(u_2), u) \leq \lambda \|u\|^2. \quad (5.4)$$

Then, recalling (4.8) and (4.10) and using monotonicity of f , we arrive at

$$\frac{1}{2} \frac{d}{dt} \left(\|u\|_{V'}^2 + \varepsilon \|u\|^2 \right) + \frac{1}{2} \|\nabla u\|^2 \leq \lambda \|u\|^2. \quad (5.5)$$

Noting that, by the Poincaré-Wirtinger inequality,

$$\lambda \|u\|^2 \leq \frac{1}{4} \|\nabla u\|^2 + c \|u\|_{V'}^2, \quad (5.6)$$

we can integrate (5.5) over (τ, T) for $\tau > 0$. Using Gronwall’s lemma, we obtain

$$\|u_1 - u_2\|_{L^\infty(\tau, T; V')}^2 \leq c(T) \|u_1(\tau) - u_2(\tau)\|_{V'}^2, \quad (5.7)$$

where $c(T)$ is independent of τ . Then, uniqueness follows by taking the limit $\tau \searrow 0$ and owing to continuity of weak solutions with values in V' (which is an obvious consequence of (2.24)).

Let us now switch to existence. To start with, we approximate the initial datum u_0 as specified in Lemma 3.1. Then, thanks to Corollary 3.3, for any $\delta \in (0, 1/6)$, there exists a “classical” solution (u_δ, w_δ) to Problem (P) defined *at least* on the time interval $(0, T_\delta)$, where T_δ depends on u_0 and δ . Actually, in principle, we may have that $T_\delta \searrow 0$ as we let $\delta \searrow 0$. On the other hand, the forthcoming argument will exclude this eventuality and show that, in fact, (u_δ, w_δ) can be extended up to the final time T .

Indeed, let us denote as $T_{\delta, \max}$ the maximum time up to which (u_δ, w_δ) can be extended in the form of a “classical” solution; namely,

$$T_{\delta, \max} := \sup \{ S \in (0, T] : u_\delta \text{ admits a “classical” extension over } (0, S) \}. \quad (5.8)$$

Due to uniqueness proved above, all extensions of (u_δ, w_δ) can be “glued” together. Consequently, there exists a (unique) maximal classical extension $(u_{\delta, \max}, w_{\delta, \max})$ defined over $(0, T_{\delta, \max})$. We claim that $T_{\delta, \max} = T$, and, to prove this claim, we proceed as usual by contradiction. Actually, due to (2.18)-(2.19) and (2.23), for any $S \in (0, T_{\delta, \max})$ we have

$$\|u_{\delta, \max}\|_{H^1(0, S; V) \cap W^{1, \infty}(0, S; V') \cap L^\infty(0, S; H^2(\Omega))} \leq C(S), \quad \varepsilon \|u_{\delta, \max}\|_{W^{1, \infty}(0, S; H)} \leq C(S), \quad (5.9)$$

$$\|w_{\delta, \max}\|_{L^\infty(0, S; V) \cap L^2(0, S; H^3(\Omega))} \leq C(S), \quad \varepsilon \|w_{\delta, \max}\|_{L^\infty(0, S; H^2(\Omega))} \leq C(S), \quad (5.10)$$

$$-1 + \epsilon(S) \leq u_{\delta, \max}(t, x) \leq 1 - \epsilon(S) \quad \forall (t, x) \in [0, S] \times \overline{\Omega}, \quad (5.11)$$

where $C(S), \epsilon(S) > 0$ and it may be $C(S) \nearrow \infty$ and $\epsilon(S) \searrow 0$ as $S \nearrow T_{\delta, \max}$. On the other hand, since $(u_{\delta, \max}, w_{\delta, \max})$ is a “classical” solution, it satisfies the a-priori estimates of the previous section on the time interval $(0, T_{\delta, \max})$. Then, thanks to (4.62), we have that

$$-1 + \bar{\epsilon} \leq u_{\delta, \max}(t, x) \leq 1 - \bar{\epsilon} \quad \forall (t, x) \in [\tau, S] \times \bar{\Omega}, \quad (5.12)$$

and for all $0 < \tau < S < T_{\delta, \max}$, with $\bar{\epsilon}$ independent both of S and of δ . To be more precise, we have that

$$\bar{\epsilon}^{-1} = Q(\mathcal{E}(u_{0, \delta}), \tau^{-1}) \leq Q(\mathbb{E}_0, \tau^{-1}), \quad (5.13)$$

where the second inequality is a consequence of (3.3).

Analogously, we have estimates of the norms in (5.9)-(5.10) over the time interval (τ, S) by a constant C independent of δ and S . Consequently, we obtain

$$\exists \bar{u} = \lim_{t \nearrow T_{\delta, \max}} u_{\delta, \max}(t, \cdot). \quad (5.14)$$

To be more precise, this limit is reached in the weak topology of $H^2(\Omega)$. Indeed, it is a consequence of (5.9) that $u_{\delta, \max} \in C_w([0, T_{\delta, \max}]; H^2(\Omega))$. Thus, $-1 + \bar{\epsilon} \leq \bar{u}(x) \leq 1 - \bar{\epsilon}$ for all $x \in \bar{\Omega}$ and we can use \bar{u} as a new “initial” datum and extend the solution $(u_{\delta, \max}, w_{\delta, \max})$ beyond the time $T_{\delta, \max}$. Moreover, the extension is still a classical solution since \bar{u} is Hölder continuous and uniformly separated from -1 and 1 . This contradicts the maximality of $T_{\delta, \max}$ and of $(u_{\delta, \max}, w_{\delta, \max})$. Hence, we necessarily have that $T_{\delta, \max} = T$.

To conclude the proof, we need to show that we can take the limit $\delta \searrow 0$ and obtain a weak solution to Problem (P). With this purpose we rename simply as (u_δ, w_δ) the maximal solution obtained in the previous part (which is now defined in the whole $(0, T)$), and observe that, thanks to estimates (4.1), (4.4), (4.29), and (4.22), there hold the following convergence relations:

$$u_\delta \rightarrow u \quad \text{weakly star in } H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^\infty((0, T) \times \Omega), \quad (5.15)$$

$$\varepsilon u_\delta \rightarrow \varepsilon u \quad \text{weakly in } H^1(0, T; H), \quad (5.16)$$

$$w_\delta \rightarrow w \quad \text{weakly in } L^2(0, T; V), \quad (5.17)$$

$$z_\delta \rightarrow z \quad \text{weakly star in } L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \quad (5.18)$$

$$v_\delta = f(u_\delta) \rightarrow v \quad \text{weakly in } L^2(0, T; V), \quad (5.19)$$

for suitable limit functions u, w, z, v . The above properties, as well as the ones that will follow, are to be intended up to the extraction of (non-relabelled) subsequences of $\delta \searrow 0$. We then immediately see that relation (1.1) passes to the limit. However, since it is only $w \in L^2(0, T; V)$, the Laplace operator with the boundary condition $\partial_n w = 0$ have to be interpreted in the weak form through the operator A (cf. (2.1)).

Next, applying the Aubin-Lions lemma, (5.15) gives

$$u_\delta \rightarrow u \quad \text{strongly in } L^p((0, T) \times \Omega), \quad \forall p \in [1, \infty). \quad (5.20)$$

Thus, a standard monotonicity argument (see, e.g., [4, Prop. 1.1, p. 42]) permits to infer from (5.19) that $v = f(u)$ almost everywhere. Moreover, by the generalized Lebesgue’s theorem, we get more precisely

$$v_\delta = f(u_\delta) \rightarrow v = f(u) \quad \text{strongly in } L^p((0, T) \times \Omega), \quad \forall p \in [1, 2). \quad (5.21)$$

Now, as a consequence of estimate (4.77), we obtain

$$\nabla v_\delta \rightarrow \nabla v \quad \text{weakly in } L^p((0, T) \times \Omega), \quad \text{for some } p > 2. \quad (5.22)$$

Collecting (5.20) and (5.22), we infer

$$\frac{u_\delta}{2} |\nabla v_\delta|^2 \rightarrow \Phi \quad \text{weakly in } L^p((0, T) \times \Omega), \quad \text{for some } p > 1, \quad (5.23)$$

and for a suitable limit function Φ .

We can now write equation (2.17) for the solution (u_δ, w_δ) and see that, thanks to the above convergence relations, all terms pass to the limit. Actually, what we obtain for $\delta \searrow 0$ is

$$w = -\Delta v + v + \Phi - \lambda u + \varepsilon u_t, \quad (5.24)$$

at least in the distributional sense. To conclude the proof, we need to identify the function Φ . To this aim, we notice that, by (5.15), (5.17), (5.23), and a comparison of terms in (2.29),

$$\| -\Delta v_\delta \|_{L^p((0,T) \times \Omega)} \leq c, \quad \text{for some } p > 1. \quad (5.25)$$

Thus, we get that, also in the limit, $-\Delta v$ (the distributional Laplacean) lies in $L^p((0,T) \times \Omega)$ for some $p > 1$. More precisely, thanks to the no-flux condition and to elliptic regularity, we deduce

$$v_\delta \rightarrow v \quad \text{weakly in } L^p(0,T; W^{2,p}(\Omega)), \quad \text{for some } p > 1, \quad (5.26)$$

and $\partial_n v = 0$ on $(0,T) \times \Gamma$ in the sense of traces. Coupling (5.21) and (5.26) we obtain strong convergence of ∇v_δ by interpolation. Indeed, we can use (for example) the Gagliardo-Nirenberg inequality (see again [22]) in the form

$$\| \nabla(v_\delta - v) \|_{L^{24/19}(\Omega)} \leq c \| D^2(v_\delta - v) \|_{L^1(\Omega)}^{1/2} \| v_\delta - v \|_{L^{12/7}(\Omega)}^{1/2} + \| v_\delta - v \|_{L^1(\Omega)}. \quad (5.27)$$

Then, integrating in time, and using the boundedness of $W^{2,p}$ -norms resulting from (5.26), we infer

$$\begin{aligned} \| \nabla(v_\delta - v) \|_{L^1(0,T; L^{24/19}(\Omega))} &\leq c \| D^2(v_\delta - v) \|_{L^1((0,T) \times \Omega)}^{1/2} \| v_\delta - v \|_{L^1(0,T; L^{12/7}(\Omega))}^{1/2} + \| v_\delta - v \|_{L^1((0,T) \times \Omega)} \\ &\leq c (\| D^2 v_\delta \|_{L^1((0,T) \times \Omega)}^{1/2} + \| D^2 v \|_{L^1((0,T) \times \Omega)}^{1/2}) \| v_\delta - v \|_{L^1(0,T; L^{12/7}(\Omega))}^{1/2} + \| v_\delta - v \|_{L^1((0,T) \times \Omega)} \\ &\leq c \| v_\delta - v \|_{L^1(0,T; L^{12/7}(\Omega))}^{1/2} + \| v_\delta - v \|_{L^1((0,T) \times \Omega)}. \end{aligned} \quad (5.28)$$

Owing to (5.21), we then obtain that ∇v_δ tends to ∇v , say, strongly in $L^1((0,T) \times \Omega)$, and consequently almost everywhere. Thus, recalling (5.22) and using once more the generalized Lebesgue theorem, we have

$$\nabla v_\delta \rightarrow \nabla v \quad \text{strongly in } L^p((0,T) \times \Omega), \quad \text{for some } p > 2. \quad (5.29)$$

whence, by (5.20), (5.23) is improved up to

$$\frac{u_\delta}{2} |\nabla v_\delta|^2 \rightarrow \frac{u}{2} |\nabla v|^2 \quad \text{strongly in } L^p((0,T) \times \Omega), \quad \text{for some } p > 1. \quad (5.30)$$

Consequently, the function Φ in (5.24) is identified to its expected limit. Hence, we get (2.29) (holding as a relation in $L^p(\Omega)$ for some $p > 1$, hence almost everywhere) and the boundary condition (2.31). Finally, we notice that the limit $\delta \searrow 0$ can be taken trivially in the initial condition (1.3). This concludes the proof.

Remark 5.1. Theorem 2.4 states that uniqueness holds for weak solutions that are classical on all intervals (τ, T) , $\tau > 0$. This is, in fact, the same regularity class where we are able to prove existence. However, we cannot exclude that uniqueness might instead fail as one considers the (larger) class of *all* weak solutions, which in particular may contain some trajectory that does not achieve the “classical” regularity for strictly positive times. Actually, some of the calculations given in the proof (in particular, those related to the gradient terms) seem not be justified under the sole regularity conditions proper of weak solutions (which, for instance, may not be “separated” from the singular values ± 1).

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